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# On the Ternary Approach to Clifford Structures and Ising Lattices

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*In memory of Professor Jaime Keller*

**Abstract.** We continue to modify and simplify the Ising-Onsager-Zhang procedure for analyzing simple orthorhombic Ising lattices by considering some fractal structures in connection with Jordan and Clifford algebras and by following Jordan-von Neumann-Wigner (JNW) approach. We concentrate on duality of complete and perfect JNW-systems, in particular ternary systems, analyze algebras of complete JNW-systems, and prove that in the case of a composition algebra we have a self-dual perfect JNW-system related to quaternion or octonion algebras. In this context, we are interested in the product table of the sedenion algebra.

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**Keywords.** Clifford algebra, crystal lattice, Ising lattice, Jordan algebra, octonions, quaternions, sedenions.

## 1. Introduction and the organization of the paper

After recalling Jordan-von-Neumann-Wigner (JNW) procedure and Ising-Onsager-Zhang (IOZ) approach, we discuss a JNW-system and related fractal structures. Then, we turn our attention to the duality structure of complete and perfect ternary systems. We consider algebras of JNW-systems, in particular algebras of Clifford type. We conclude with a study of a composition algebra and self-dual perfect JNW-systems. Continuing our previous papers [10, 11, 13] we prove in this case it is one of quaternion and octonion algebras, and this needs a study of the product table of sedenion algebra. The idea of applying the JNW-systems, considered in Section 3 and thereafter, to binary and ternary alloys may be to some extent illustrated for instance by the first step of fractal construction related to an  $AB_3$  binary alloy of an fcc lattice and (111) surface orientation [2, Fig. 1].

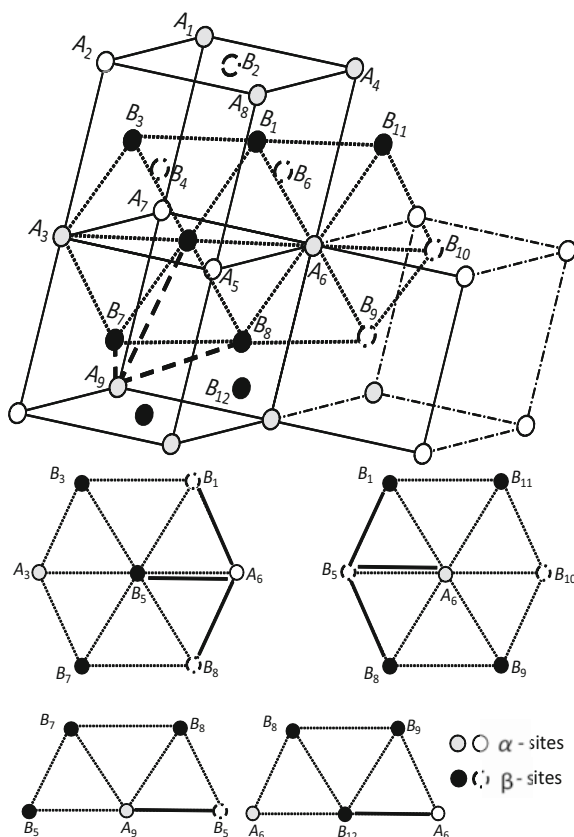
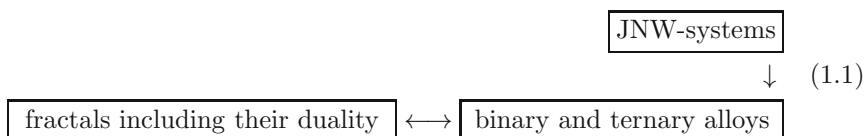
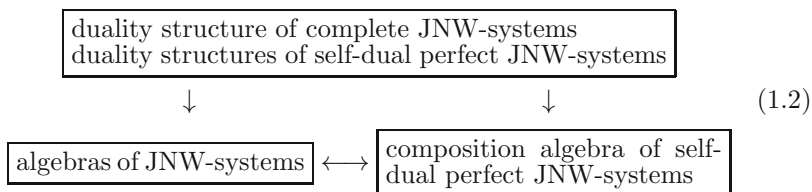


FIGURE 1.  $AB_3$  binary alloy of an fcc lattice and (111) surface orientation.

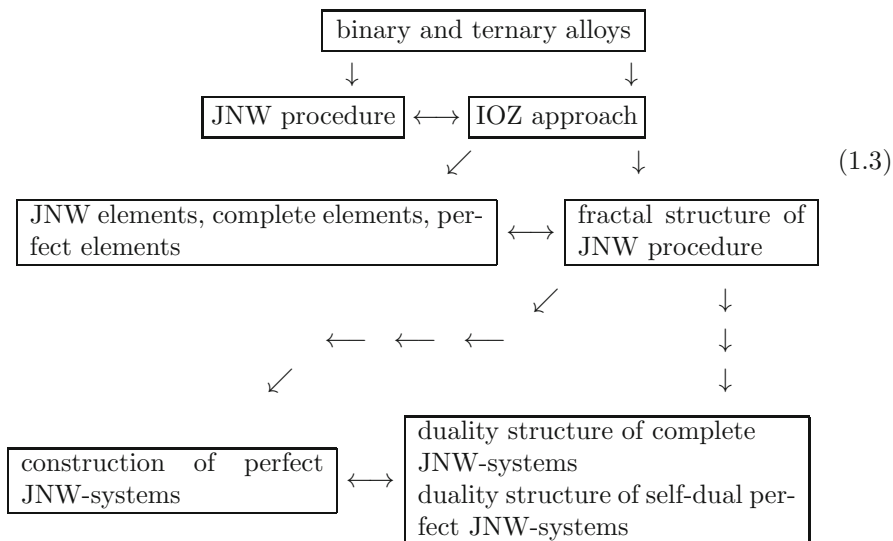
The choice in this first step of the fractal construction is tightly related to the theory of cooperative phenomena of Kikuchi [9]. From our point of view it is important that like in the case of modern building model constructions, Kikuchi takes into account collections of adjacent edges, mostly three or four, of the crystal lattice. This suggests our subsequent steps: duality of fractals, duality fractals, JNW-systems, duality structure of complete JNW-systems, composition algebras of self-dual perfect JNW-systems, etc. These ideas can also be expressed by the following scheme:



The content of the paper may be visualized as follows:



In order to pass from (1.1) to (1.2), we need the following expository (historical) part of the paper:



## 2. Jordan-von Neumann-Wigner procedure

Jordan, von Neumann, and Wigner proved [7] (cf. also [4–6]) that among irreducible algebras there are algebras  $\mathbb{H}_q(\mathbb{F})$  of Hermitian  $q \times q$ -matrices with entries in a field  $\mathbb{F}$ , namely

$$\mathbb{H}_3(\mathbb{O}) \text{ and } \mathbb{H}_q(\mathbb{F}) \text{ with } \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H} \text{ and } q > 2. \quad (2.1)$$

As it was observed in [11, p. 737], and then repeated in [21, p. 7], the natural appearance of multiplication

$$A \circ B = \frac{1}{2}(AB + BA) \quad (2.2)$$

instead of the usual matrix multiplication replaces in elegant way the desire of Ising [3], Onsager [16], Zhang [20], and Staples [18] of finding commutative subalgebras of the given algebra and studying their properties related to the superposition of bilinear forms.

### 3. Ising-Onsager-Zhang approach

The models of Ising [3] and Onsager [16] are extended by Zhang [20] so that the resulting model applies to a simple orthorhombic crystal lattice in a three-dimensional space. This extension involves, in particular, opening knots by a rotation in a higher dimensional space, and studying important commutators in the corresponding algebra.

In our approach, we extend Zhang's idea of decomposing knots related to lattice vertices to deformations of formal knot sequences arising from Reidemeister moves [19]. The model consisting of an orthorhombic lattice constructed by  $m$  rows and  $n$  sites per row in one of the  $\ell$  planes involves [11]  $2^{n \cdot \ell}$ -dimensional matrices  $s_{r,s}^\alpha$ ,  $\alpha = 1, 2, 3$ ,  $1 \leq s \leq \ell$ ,  $1 \leq r \leq n$ , related to the so-called *split-quaternions* and, correspondingly, to the familiar Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

namely:

$$\begin{aligned} s_{r,s}^1 &= \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{1} \otimes \sigma_1 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{1}, \\ s_{r,s}^2 &= \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{1} \otimes \frac{1}{i} \sigma_2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathbf{1} \quad \text{with} \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ s_{r,s}^3 &= \mathbf{1} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{r-2, s-2} \otimes \sigma_3 \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{n-r, \ell-s}. \end{aligned}$$

It is worthwhile to notice that both Onsager and Zhang analyze continued fractions and nested series, e.g.,

$$\frac{\sqrt{5}-1}{2} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{1 - \sqrt{1 - \cdots}}}}} \quad (3.1)$$

which give rise to two kinds of fractals: of flower type and of branch type, and to the problem of their duality [8, 10]. The objects of type (3.1) appear as a result of calculations in [16] and [20], but they inspire in us a relationship with the two types of fractals mentioned.

### 4. Jordan-von Neumann-Wigner system

We begin by introducing two kinds of systems for a set

$$\{e_1, e_2, \dots, e_n\}. \quad (4.1)$$

The first system is called a *triad*. It is obtained by choosing three distinct elements from the set (4.1), for example,  $(e_1, e_2, e_3)$ . The second system is our JNW-system: we choose two elements, for example,  $(e_i, e_j)$ . Then, we select the third element, for example,  $e_k \notin \{e_i, e_j\}$ , and by doing so we define a JNW-element: the triple  $(e_i, e_j, e_k)$ . The most important elements arise from

the binary product of an algebra. All JNW-elements selected from (4.1) make up a JNW-system.

**Definition 4.1.** A JNW-system consisting exclusively of complete JNW-elements is called *complete* if it satisfies the following conditions:

- (a) For any pair of elements  $(e_i, e_j)$  we can find a unique element  $e_k$  which constitutes a JNW-element  $(e_i, e_j, e_k)$ .
- (b) For any element  $e_k$  we can find a (not necessarily unique) pair  $(e_i, e_j)$  which constitutes a JNW-element  $(e_i, e_j, e_k)$ .
- (c) Each element is symmetric, i.e.,

$$(e_i, e_j, e_k) = (e_{\sigma(i)}, e_{\sigma(j)}, e_{\sigma(k)})$$

for any permutation  $\sigma$  of the three objects considered.

**Definition 4.2.** A JNW-system consisting exclusively of perfect JNW-elements is called *perfect* if it satisfies conditions (a)–(c) from Definition 4.1 with the condition (b) replaced by:

- (d) For any element  $e_k$  we can find pairs  $(e_i, e_j)$  such that each triple  $(e_i, e_j, e_k)$  is a JNW-element and where the number of pairs  $(e_i, e_j)$  does not depend on the given  $e_k$ .

## 5. Fractal structure of Jordan-von Neumann-Wigner procedure

We recall the following result from [11].

**Theorem A.** Let  $(e_1, e_2, \dots, e_n)$  be an  $n$ -sequence.

- (A1) The sequence generates a complete JNW-system if and only if  $n = (2l + 1)2^k - 1$ ,  $l = 1, 2, \dots$ ,  $k = 2, 3, \dots$ ;
- (A2) The sequence generates a perfect JNW-system if and only if  $n = 2^k - 1$ ,  $k = 2, 3, \dots$ .

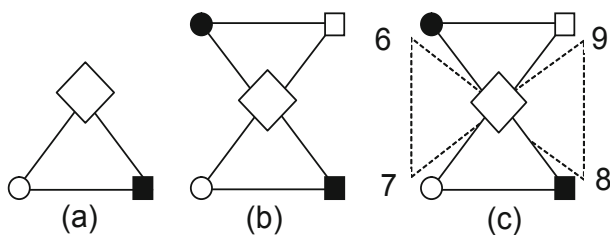


FIGURE 2. An example of construction of simple complete JNW-systems.

## 6. Construction of a perfect JNW-system

As in [11], we construct a base system of ternaries  $Z(n)$  and generate the total system. For  $n = 3$  we take  $(e_1, e_2, e_3)$  and let  $Z(3) = (1, 2, 3)$ . Next, we choose one element from among  $e_1, e_2, e_3$  which we call a *polar point*. Here, for example, we choose  $e_1$ , therefore we consider elements  $(e_1, e_2, e_3)$ ,  $(e_1, e_4, e_6)$  and  $(e_1, e_5, e_7)$ . The later step, involving  $e_4, e_5, \dots$  does not apply in the case of  $n = 3$ .

Then, we fix the JNW-system:

$$Z(15) = (2, 3) *_1 (4, 6) *_1 (5, 7) *_1 (8, 10) *_1 (9, 11) *_1 (12, 14) *_1 (13, 15).$$

The relation  $*_1$ , connected with the vertex  $e_1$ , is visualized in Fig. 3.

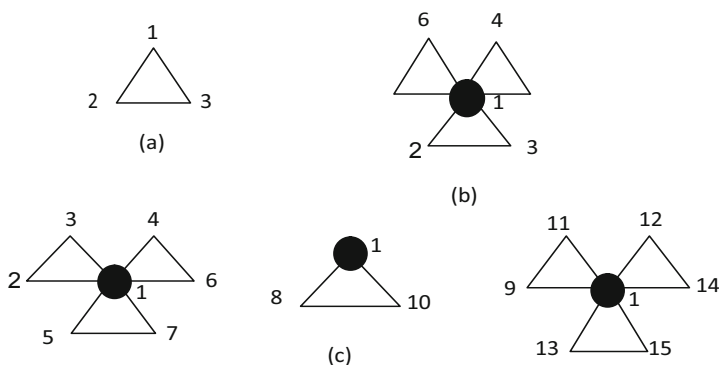


FIGURE 3. An example of construction of simple perfect JNW-systems.

Repeating this process and making a network, understood as a connected system of non intersecting triangles, such that each pair of neighboring element has a common vertex or common site, we can obtain a complete set of all perfect JNW-systems, in question.

It is clear that the structure in question is perfectly suitable for investigating ternary alloys without vacancies, binary, as well as ternary alloys with vacancies (cf. the triangle  $\triangle(1, 8, 10)$  in Fig. 3(c)).

## 7. Duality structure of complete JNW-systems and the characterization of self-dual perfect JNW-systems

In this section we shall determine a duality structure for complete ternary systems and introduce a concept of a self-dual perfect JNW-system which characterizes the composition algebra directly. At first, we describe the duality structure for a complete JNW-system.

### 7.1. The duality structure of complete ternary systems

We discuss the duality structure of complete ternary systems. By use of the basic construction of complete ternary systems, let us define

$$\Lambda : \Lambda(1) = 1, \quad \Lambda(t) = 2t, \quad 2 \leq t \leq 2^{k-1}.$$

Hence, we have a mapping called a *duality mapping of genetic type*, cf. [12]. We refer to each of the following three sets

$$\{1\}, \quad \{t : 2 \leq t \leq 2^{k-1}\}, \quad \{\Lambda(t) : 2 \leq t \leq 2^k\}$$

as a *branch of the system*. A ternary system is called *self dual* when each branch of the system is preserved under the duality mapping.

### 7.2. The dual structure of perfect ternary systems

We are going to discuss the duality structure of perfect ternary systems:

$$\Lambda : \Lambda(2) = 2, \quad \Lambda(2^{k-1}) = 2^{k-1} + 2, \quad k > 2; \quad \Lambda(t) = 2t, \quad 3 \leq t < 2^{k-1}.$$

Hence, we have the mapping which is natural to call *duality of JNW-type*. Moreover, we have the following theorem.

#### Theorem 7.1.

- (T1) *Each complete JNW-system is self dual.*  
 (T2) *A perfect JNW-system is self dual if and only if  $n = 3$  or  $n = 7$ .*

## 8. Algebras of JNW-systems

We are going to introduce algebras of a JNW-system. Here we restrict ourselves to a special class of the following unital real (not necessarily associative) algebras.

**Definition 8.1.** A finitely generated algebra  $A = \mathbb{R}[e_1, e_2, \dots, e_n]$  is said to be of *monic product type* if, for any two generators  $e_i, e_j \in A$ , we have  $e_i e_j = \eta_{ij} e_k$  for some generator  $e_k$  and  $\eta_{ij} \in \mathbb{R}$ .

We concentrate on algebras of a perfect JNW-system. We begin with the following definition.

**Definition 8.2.** A finitely generated algebra  $A = \mathbb{R}[e_1, e_2, \dots, e_n]$  is said to be of *Clifford type* if the following commutation relations are satisfied:

$$e_i e_j + e_j e_i = 0 \quad \text{for all } i \neq j \text{ and } i, j \neq 1.$$

In the case when  $A$  is an *associative*, it becomes a *Clifford algebra* when it additionally satisfies the following condition:

$$e_j^2 = \eta_j e_i \quad \text{for some } i \text{ and } \eta_j \in \mathbb{R}.$$

The following theorem has been proven in [11]:

**Theorem B.**

- (B1) Let  $A$  be a finitely generated algebra  $A = \mathbb{R}[e_1, e_2, \dots, e_n]$  of Clifford type. Then the JNW-system of  $A$  is perfect.
- (B2) Conversely, choosing a perfect system, we can find an algebra of Clifford type which realizes the system.

We illustrate Theorem B with a few examples.

*Example 1.* Mendel algebra:

	$e_1$	$e_2$
$e_1$	$2e_1$	$e_2$
$e_2$	$e_2$	0

*Example 2.* Algebra of complex numbers:

	$e_1$	$e_2$
$e_1$	$e_1$	$e_2$
$e_2$	$e_2$	$-e_1$

*Example 3.* Quaternion algebra:

	$e_1$	$e_2$	$e_3$	$e_4$
$e_1$	$e_1$	$e_2$	$e_3$	$e_4$
$e_2$	$e_2$	$-e_1$	$e_4$	$-e_3$
$e_3$	$e_3$	$-e_4$	$-e_1$	$e_2$
$e_4$	$e_4$	$e_3$	$-e_2$	$-e_1$

*Example 4.* Octonion algebra ( $\varepsilon = 1$  in the table below):

	$e_0$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_0$	$e_0$	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	$e_1$	$-e_0$	$e_3$	$-e_2$	$e_5$	$-e_4$	$-\varepsilon e_7$	$\varepsilon e_6$
$e_2$	$e_2$	$-e_3$	$-e_0$	$e_1$	$e_6$	$\varepsilon e_7$	$-e_4$	$-\varepsilon e_5$
$e_3$	$e_3$	$e_2$	$-e_1$	$-e_0$	$e_7$	$-\varepsilon e_6$	$\varepsilon e_5$	$-e_4$
$e_4$	$e_4$	$-e_5$	$-e_6$	$-e_7$	$-e_0$	$\varepsilon e_1$	$\varepsilon e_2$	$e_3$
$e_5$	$e_5$	$e_4$	$-\varepsilon e_7$	$-e_6$	$-\varepsilon e_1$	$-e_0$	$-\varepsilon e_3$	$\varepsilon e_2$
$e_6$	$e_6$	$\varepsilon e_7$	$e_4$	$-\varepsilon e_5$	$-\varepsilon e_2$	$\varepsilon e_3$	$-e_0$	$-\varepsilon e_1$
$e_7$	$e_7$	$-\varepsilon e_6$	$e_5$	$e_4$	$-e_3$	$-\varepsilon e_2$	$\varepsilon e_1$	$-e_0$

*Example 5.* See the sedenion algebra defined in Table 5.

## 9. Composition algebra and self-dual perfect JNW-systems

Finally, we characterize composition algebras in terms of the JNW-systems. At first, we recall the definition of a composition algebra.

**Definition 9.1.** A unital algebra  $A$  over  $\mathbb{R}$  with a nondegenerate norm  $N : A \rightarrow \mathbb{R}^+$  is a *composition algebra* if  $N(XY) = N(X)N(Y)$  for any two elements  $X, Y \in A$ .



	1	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$	$e_8$	$e_9$	$e_{10}$	$e_{11}$	$e_{12}$	$e_{13}$	$e_{14}$	$e_{15}$
1	1	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$	$e_8$	$e_9$	$e_{10}$	$e_{11}$	$e_{12}$	$e_{13}$	$e_{14}$	$e_{15}$
$e_1$	$e_1$	-1	$e_3$	- $e_2$	$e_5$	- $e_4$	- $e_7$	$e_6$	$e_9$	- $e_8$	- $e_{11}$	$e_{10}$	- $e_{13}$	$e_{12}$	$e_{15}$	- $e_{14}$
$e_2$	$e_2$	- $e_3$	-1	$e_1$	$e_6$	$e_7$	- $e_4$	- $e_5$	- $e_{10}$	$e_{11}$	- $e_8$	- $e_9$	- $e_{14}$	- $e_{15}$	$e_{12}$	$e_{13}$
$e_3$	$e_3$	$e_2$	- $e_1$	-1	$e_7$	- $e_6$	$e_5$	- $e_4$	$e_{11}$	- $e_{10}$	$e_9$	- $e_8$	- $e_{15}$	$e_{14}$	- $e_{13}$	$e_{12}$
$e_4$	$e_4$	- $e_5$	- $e_6$	- $e_7$	-1	$e_1$	$e_2$	$e_3$	$e_{12}$	$e_{13}$	$e_{14}$	$e_{15}$	- $e_8$	- $e_9$	- $e_{10}$	- $e_{11}$
$e_5$	$e_5$	$e_4$	- $e_7$	$e_6$	- $e_1$	-1	- $e_3$	$e_2$	$e_{13}$	- $e_{12}$	$e_{15}$	- $e_{14}$	$e_9$	- $e_8$	$e_{11}$	- $e_{10}$
$e_6$	$e_6$	$e_7$	$e_4$	- $e_5$	- $e_2$	$e_3$	-1	- $e_1$	$e_{14}$	- $e_{15}$	- $e_{12}$	$e_{13}$	$e_{10}$	- $e_{11}$	- $e_8$	$e_9$
$e_7$	$e_7$	- $e_6$	$e_5$	$e_4$	- $e_3$	- $e_2$	$e_1$	-1	$e_{15}$	$e_{14}$	- $e_{13}$	- $e_{12}$	$e_{11}$	$e_{10}$	- $e_9$	- $e_8$
$e_8$	$e_8$	- $e_9$	- $e_{10}$	- $e_{11}$	- $e_{12}$	- $e_{13}$	- $e_{14}$	- $e_{15}$	-1	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_9$	$e_9$	$e_8$	- $e_{11}$	$e_{10}$	- $e_{13}$	$e_{12}$	$e_{15}$	- $e_{14}$	- $e_1$	-1	- $e_3$	$e_2$	- $e_5$	$e_4$	$e_7$	- $e_6$
$e_{10}$	$e_{10}$	$e_{11}$	$e_8$	- $e_9$	- $e_{14}$	- $e_{15}$	$e_{12}$	$e_{13}$	- $e_2$	$e_3$	-1	- $e_1$	- $e_6$	- $e_7$	$e_4$	$e_5$
$e_{11}$	$e_{11}$	- $e_{10}$	$e_9$	$e_8$	- $e_{15}$	$e_{14}$	- $e_{13}$	$e_{12}$	- $e_3$	- $e_2$	$e_1$	-1	- $e_7$	$e_6$	- $e_5$	$e_4$
$e_{12}$	$e_{12}$	$e_{13}$	$e_{14}$	$e_{15}$	$e_8$	- $e_9$	- $e_{10}$	- $e_{11}$	- $e_4$	$e_5$	$e_6$	$e_7$	-1	- $e_1$	- $e_2$	- $e_3$
$e_{13}$	$e_{13}$	- $e_{12}$	$e_{15}$	- $e_{14}$	$e_9$	$e_8$	$e_{11}$	- $e_{10}$	- $e_5$	- $e_4$	$e_7$	- $e_6$	$e_1$	-1	$e_3$	- $e_2$
$e_{14}$	$e_{14}$	- $e_{15}$	- $e_{12}$	$e_{13}$	$e_{10}$	- $e_{11}$	$e_8$	$e_9$	- $e_6$	- $e_7$	- $e_4$	$e_5$	$e_2$	- $e_3$	-1	$e_1$
$e_{15}$	$e_{15}$	$e_{14}$	- $e_{13}$	- $e_{12}$	$e_{11}$	$e_{10}$	- $e_9$	$e_8$	- $e_7$	$e_6$	- $e_5$	- $e_4$	$e_3$	$e_2$	- $e_1$	-1

TABLE 1. Sedenion algebra

**Definition 9.2.** An algebra  $A$  with a norm  $N : A \rightarrow \mathbb{R}^+$  is said to be *ternary of composition type* if its two ternaries  $(e_i, e_j, e_k)$  and  $(e_r, e_s, e_t)$  have a common vertex, for example  $e_k = e_t$ . When studying the commutation relations we may use the notation  $(e_i, e_j : e_k)$ , etc..

We know from the Hurwitz's Theorem that every (unital) composition algebra (over the reals) which, at the same time, is a normed division algebra, is isomorphic to one of the following four algebras: the real numbers  $\mathbb{R}$ , the complex numbers  $\mathbb{C}$ , the quaternions  $\mathbb{H}$ , or the Cayley numbers (the octonions)  $\mathbb{O}$ .

Here we shall treat this characterization in terms of the perfect JNW-systems. We consider the characterization of the problem under the assumption that the algebra is of monic product type. We have the following theorem.

**Theorem 9.3.**

- (T3) *If  $A$  is an alternative algebra, then we have a perfect JNW-system.*  
 (T4) *Moreover, if  $A$  is an alternative algebra, then the related perfect JNW-system is self dual. Hence,  $A$  is either the quaternion algebra or the octonion algebra.*

*Remark 9.4.* Originally the authors formulated Theorem 9.3 for a *composition algebra*. However, by the results of Guy Roos [17] we may suppose that  $A$  is an *alternative algebra*. Precisely, in [17, p. 162], a relation is shown for an algebra  $A$  with a unit element  $e$  and an involutive anti-automorphism  $a \mapsto a^*$  such that  $a + a^*$  and  $aa^*$  are spanned by  $e$  for all  $a$  in  $A$ . With the notation  $n(a) = aa^*$ , if  $n$  is a nonsingular mapping into the field of  $A$  and  $A$  is alternative, then  $(A, n)$  is a composition algebra [1].

*Proof of (T3).* We notice that we have the following identities from the assumption that  $A$  is an alternative algebra:

$$X(YX) = (XY)X, \quad (9.1)$$

$$(XX)Y = X(XY), \quad (9.2)$$

$$X(YY) = (XY)Y, \quad (9.3)$$

for every  $X, Y \in A$ .

Then, letting  $X = \sum \alpha_i e_i$  and  $Y = \sum \beta_j e_j$ , we find the following equalities from, respectively, (9.1), (9.2), and (9.3):

$$(e_i, e_j : e_k) + (e_k, e_j : e_i) = (e_i : e_j, e_k), \quad (9.4)$$

$$(e_i, e_j : e_k) + (e_j, e_i : e_k) = (e_i : e_j, e_k) + (e_j : e_i, e_k), \quad (9.5)$$

$$(e_i, e_j : e_k) + (e_i, e_k : e_j) = (e_i, e_j : e_k) + (e_i : e_k, e_j). \quad (9.6)$$

From these equalities we can see the symmetry condition for elements and conclude that our JNW-system is perfect.  $\square$

*Proof of (T4).* Here we only show that we can find elements of the sedenion algebra  $S = \mathbb{R}[1, e_1, \dots, e_{15}]$  which do not satisfy the condition of composition algebras. We choose two systems:  $(e_8, e_{10}, e_2)$  and  $(e_{12}, e_{14}, e_2)$ . Choosing

$X = e_3 + e_{10}$  and  $Y = e_6 - e_{15}$ , we can see that  $XY = 0$ . Here we have chosen the following commutation relations:

$$e_3e_6 = e_5, \quad e_{10}e_6 = e_{12}, \quad e_3e_{15} = e_{12}, \quad e_{10}e_{15} = e_5. \quad (9.7)$$

Thus, we can see that the condition may not be satisfied for non-self dual perfect systems. In order to prove the assertion, we realize the system by use of the realization theorem, reduce the problem to sedenions, and have to prove it. The proof follows from the relations (9.7) and the table of Example 5.  $\square$

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